ON FINITE DIFFERENTIAL TROPICAL BASIS TO LINEAR ODE'S

STEFANO MERETA AND ALEJANDRO VARGAS

Abstract. We formulate several open questions regarding the tropicalization of linear ODEs, aiming primarily to develop methods for calculating the radius of convergence of their classical solutions. To this aim it is of foremost importance to characterize the classes of equations that admit a finite differential tropical basis, as introduced in [FT22]. Our initial exploration examines the second- and third-order cases.

1. Introduction

Since at least the 1930s, algebraic techniques have been developed to solve (partial) differential equations expressed as polynomials in an indeterminate function and its derivatives. These efforts were carried out preeminently by J. F. Ritt and his student E. R. Kolchin (see [Rit50; Kol73]) and established *differential algebra* as a new and effective toolkit to apply commutative algebra to study (partial) differential equations and their solution sets.

More recently, *tropical geometry* (and algebra) emerged as an area of mathematics whose tools transform hard algebro-geometric problems into combinatorial ones. These are more amenable for computational approaches.

In the last ten years, since [Gri17], the use of tropical techniques for differential equations has been explored and its scope extended to differential equations with power series coefficients over arbitrary valued field (see [GM24]). While [AGT16; FGH+20; MG23] provide a proof of a fundamental theorem analogous to that of tropical geometry (see [MS15, Theorem 3.2.5]) in full generality, the computational effectiveness of these techniques remains largely unexplored.

The finiteness of tropical bases (see [MS15, Theorem 2.6.5]) is indispensable for implementing effective tropical algorithms in the classic algebro-geometric setting. When dealing with ideals generated by linear forms a tropical basis can always be chosen to consist of linear forms: the circuits of the valuated matroid associated to the linear space. Recently, [FT22] introduced a notion of *differential tropical basis* (DTB), where finiteness is defined up to the action of the derivative. However, the authors also demonstrate a class of differential ideals generated by linear differential equations with constant coefficients for which there is no finite DTB of linear forms with respect to the trivial valuation.

This work explores further the finiteness of DTBs, taking the first steps towards characterizing this property for linear differential equations in the more general framework of non-trivial valuation. Our interest in DTB finiteness stems from its potential applications in computing the the radii of convergence of differential equations over valued fields via tropical methods (see [MG23, Section 6]).

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Here we study in detail the second order case under the hypothesis that the two roots of the characteristic polynomial have different non-zero valuation. In this case we obtain that a finite DTB for the differential ideal generated by the chosen equation is guaranteed. We conclude with some considerations in the third order case and we formulate some open questions.

1.1. **Rudiments of differential and tropical algebra.** For comprehensive sources about differential algebra, we refer the reader to [Rit50; Kol73]. A *differential ring* is a ring R equipped with a derivation, i.e. an additive map $d_R : R \to R$ such that

$$d_R(rs) = rd_R(s) + d_R(r)s$$

holds for all $r, s \in R$. All along this work, the only differential ring we will consider is the ring of power series $K[\![t]\!]$ over a field K, equipped with the usual derivation d/dt. The *differential algebra of differential polynomials* in one variable over $K[\![t]\!]$ is the $K[\![t]\!]$-algebra

$$K[\![t]\!][x^{(j)} \mid j \in \mathbb{N}]$$

equipped with the only derivation d extending d/dt and such that $dx^{(j)} = x^{(j+1)}$. We will denote this differential algebra as $K[\![t]\!]\{x\}$. An ideal $I \subseteq K[\![t]\!]\{x\}$ is *differential* if $df \in I$ for all $f \in I$. Given a differential polynomial f, the *differential ideal generated by* f is the smallest differential ideal containing f, i.e. the (algebraic) ideal generated by the family $\{d^n f \mid n \in \mathbb{N}\}$. We will denote this ideal as $[f]$.

For fundamentals of valuations and tropicalization, we refer the reader to [GKMV26, Section 1.7 and 1.8] in this volume. Hereby we briefly present the notions of tropical differential algebra required in the following, as introduced in [GM24]. For an element $A \in K[\![t]\!]$ we denote as $\mathrm{Supp}(A)$ its support. If the field K is endowed with a valuation $v_K : K \to \mathbb{T}$, then the assignment $v : K[\![t]\!] \to \mathbb{T}^{(2)}$ defined as

$$A = \sum_{i=0}^{\infty} a_i t^i \mapsto (\min \mathrm{Supp}(A), v_K(a_{\min \mathrm{Supp}(A)}))$$

is a valuation. If v_K is non-trivial v is a rank 2 valuation, otherwise it is of rank 1. The tropicalization of differential polynomials takes place via the valuation v: given a differential polynomial $f \in K[\![t]\!]\{x\}$ its tropicalization $\mathrm{trop}_v(f)$ is the element of $\mathbb{T}^{(2)}[x^{(j)} \mid j \in \mathbb{N}]$ obtained by applying v to the coefficients of f. For a differential ideal $I \subset K[\![t]\!]\{x\}$ its tropicalization $\mathrm{trop}_v(I)$ is

$$\mathrm{trop}_v(I) = \{\mathrm{trop}_v(f) \mid f \in I\}.$$

Analogous to the ring-theoretic construction, tropical power series form a semiring $\mathbb{T}[\![t]\!]$. This becomes a differential semiring (for details about them see [GM24, Section 2.4]) by endowing it with the differential $(\frac{d}{dt})_{v_K}$ defined by the rule:

$$\left(\frac{d}{dt}\right)_{v_K}(t^n) = \begin{cases} v_K(n)t^{n-1} & \text{if } n \geq 1 \\ \infty & \text{if } n = 0. \end{cases}$$

We denote the differential semiring $(\mathbb{T}[\![t]\!], \left(\frac{d}{dt}\right)_{v_K})$ as $\mathbb{T}[\![t]\!]_{v_K}$.

In contrast with classic tropical geometry, power series in $K[\![t]\!]$ are not tropicalized with respect to the same map used for equations: let $A \in K[\![t]\!]$, then its tropicalization $\mathrm{trop}(A)$ is obtained by applying v_K to its coefficients, thus obtaining an element of $\mathbb{T}[\![t]\!]_{v_K}$. We denote

this assignment $\mathrm{trop}_{v_K} : K[\![t]\!] \to \mathbb{T}[\![t]\!]_{v_K}$ and drop the subscript if the situation makes it clear. It follows from the definitions that

$$\left(\frac{d}{dt}\right)_{v_K} (\mathrm{trop}_{v_K}(A)) = \mathrm{trop}_{v_K}(dA)$$

for every $A \in K[\![t]\!]$.

Finally, in order to evaluate an element f of $\mathbb{T}^{(2)}[x^{(j)} \mid j \in \mathbb{N}]$ at a tropical power series $B \in \mathbb{T}[\![t]\!]$, we introduce the homomorphism of semiring $w : \mathbb{T}[\![t]\!]_{v_K} \to \mathbb{T}^{(2)}$ defined in a similar fashion to the valuation v:

$$B = \sum_{i=0}^{\infty} b_i t^i \mapsto (\min \mathrm{Supp}(B), b_{\min \mathrm{Supp}(B)}).$$

The evaluation of $f \in \mathbb{T}^{(2)}[x^{(j)} \mid j \in \mathbb{N}]$ at a tropical power series B is the element of $\mathbb{T}^{(2)}$ obtained by plugging $(d/dt)^j_{v_K}(B)$ for $x^{(j)}$. We say that B is a *tropical solution* to f if the resulting minimum in $\mathbb{T}^{(2)}$ is attained at least twice.

1.2. **Linear homogeneous ODEs with constant coefficients.** Let K be an algebraically closed field. Recall that a linear homogeneous ODE of order r with constant coefficients $c_i \in K$ and unknown function $x(t)$ is expressed as:

$$F = x^{(r)} + c_{r-1}x^{(r-1)} + \cdots + c_1 x^{(1)} + c_0 x = 0, \tag{1.1}$$

where the c_i are in K and $x^{(i)}$ is the derivative $\frac{d^i}{dt^i}x(t)$. The solution set $\mathcal{S}(F)$ is a linear space of dimension r, determined by the associated characteristic polynomial:

$$\chi(z) = z^r + c_{r-1}z^{r-1} + \cdots + c_1 z + c_0.$$

Let κ be the number of roots of $\chi(z)$ and let $\{\alpha_s\}_{s\in[\kappa]}$ be the set of its roots. Denote as m_s the multiplicity of α_s. The functions $\{t^{j-1}\exp(\alpha_s t) \mid j = 1, \ldots, m_s\}_{s\in[\kappa]}$, defined by the power series expansion

$$t^{j-1}\exp(\alpha_s t) = \sum_{i=0}^{\infty} \frac{\alpha_s^i}{i!} t^{i+j-1}$$

are solutions to Equation (1.1) and the set $\{t^{j-1}\exp(\alpha_s t) \mid j = 1, \ldots, m_s\}_{s\in[\kappa]}$ is a basis for $\mathcal{S}(F)$. We assume that $c_0 \neq 0$, so that 0 is not a root of $\chi(z)$, and that all the roots are pairwise distinct, i.e. $\kappa = r$ and $m_s = 1$ for all $s \in [r]$: under these hypothesis $\{\exp(\alpha_s t)\}_{s\in[r]}$ is a basis for $\mathcal{S}(F)$.

1.3. **Tropicalization of ODEs under trivial valuation.** Under the trivial valuation, the tropicalization of $\mathcal{S}(F)$ yields a matroid M over $\mathbb{N}$ [ABF+23] (see [BDKPW13] for the relevant axioms on infinite matroids). To compute M, we construct an $r \times \mathbb{N}$ matrix using the power series coefficients of our basis functions $\exp(\alpha_s t)$. The bases of M are sets $I \subset \mathbb{N}$ indexing non-zero maximal minors. Because column rescaling does not alter the underlying matroid, we factor out the $1/i!$ terms and analyze the following matrix:

$$A = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \alpha_1^3 & \alpha_1^4 & \ldots \\ 1 & \alpha_2 & \alpha_2^2 & \alpha_2^3 & \alpha_2^4 & \ldots \\ \vdots & & \vdots & & \vdots & \\ 1 & \alpha_r & \alpha_r^2 & \alpha_r^3 & \alpha_r^4 & \ldots \end{pmatrix}$$

If all maximal minors are non-zero, M is the finitary rank r uniform matroid $U_{r,\mathbb{N}}$. Over fields of characteristic zero, this generic behavior follows from the properties of generalized Vandermonde determinants.

In the non-generic case, suppose $I \in \binom{\mathbb{N}}{r}$ indexes a vanishing minor. Let $k_1 < k_2 < \cdots < k_r$ denote the elements of I. Factoring out the lowest power yields

$$\begin{vmatrix} \alpha_1^{k_1} & \alpha_1^{k_2} & \cdots & \alpha_1^{k_r} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_r^{k_1} & \alpha_r^{k_2} & \cdots & \alpha_r^{k_r} \end{vmatrix} = \alpha_1^{k_1}\alpha_2^{k_1}\cdots\alpha_r^{k_1} \begin{vmatrix} 1 & \alpha_1^{k_2-k_1} & \cdots & \alpha_1^{k_r-k_1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_r^{k_2-k_1} & \cdots & \alpha_r^{k_r-k_1} \end{vmatrix}, \tag{1.2}$$

demonstrating that shifting I preserves the vanishing of the corresponding minors (as $\alpha_i \neq 0$ for all i):

Proposition 1.1. *Let $\sigma : \mathbb{Z} \to \mathbb{Z}$ denote the shift map $i \mapsto i+1$. For any integer q such that $\sigma^q(I) \subset \mathbb{N}$, the shifted index set $\sigma^q(I)$ yields a vanishing minor if and only if I does.*

This observation motivates the following question:

Question 1.2. *Can the infinite matroid M on the ground set $\mathbb{N}$ associated with a linear homogeneous ODE of order r, be described using only a finite matroid and a group action?*

1.4. **Differential tropical basis and radius of convergence.** Following [MG24, Section 6], we define the *radius of convergence with respect to* $c \in \mathbb{R}_{>1}$ of a tropical power series $\bigoplus_{i=0}^{\infty} b_i t^i \in \mathbb{T}[\![t]\!]$ as

$$\rho_c\left(\sum_{i=0}^{\infty} b_i t^i\right) := \sup\left\{r \in [0,\infty) \mid \lim_{i\to\infty} c^{-b_i} r^i = 0\right\}.$$

Here the convention is that $c^{-b_i} = 0$ when $b_i = \infty$.

In contrast, let K be an uncountable algebraically closed field of characteristic zero equipped with a nontrivial valuation $v_K : K \to \mathbb{T}$ admitting a section. We choose $c \in \mathbb{R}_{>1}$ and define the norm $|x|_{K,c} = c^{-v_K(x)}$. The *radius of convergence* of a power series $\sum_{i=0}^{\infty} a_i t^i \in K[\![t]\!]$ is given by

$$\rho_c\left(\sum_{i=0}^{\infty} a_i t^i\right) := \sup\left\{r \in [0,\infty) \mid \lim_{i\to\infty} |a_i|_{K,c} r^i = 0\right\}.$$

A direct consequence of the main theorem in [MG24] is that computing the convergence radius commutes with tropicalization.

Proposition 1.3. *For any $F \in K[\![t]\!]\{x\}$, we have:*

$$\{\rho_c(A) \colon\ A \in \mathcal{S}(F)\} = \{\rho_c(B) \colon\ B \in \mathcal{S}(\mathrm{trop}([F]))\}. \tag{1.3}$$

The right-hand side of Equation (1.3) is tropical, and thus amenable to computation via polyhedral methods and $(\min,+)$-algebra algorithms. In particular, [MG24, Example 6.4] demonstrates that having a finite DTB at hand facilitates this computation:

Definition 1.4. Let $I \subseteq K[\![t]\!]\{x\}$ be a differential ideal. As introduced in [FT22], a *differential tropical basis* (DTB) for I is a set of differential polynomials $\{F_\lambda\}_{\lambda\in\Lambda} \subset I$ for some index set Λ such that

$$\mathcal{S}(\mathrm{trop}(I)) = \bigcap_{\lambda\in\Lambda}\bigcap_{n\in\mathbb{N}} \mathcal{S}(\mathrm{trop}(d^n F_\lambda)).$$

This notion is the differential analogue to the classic notion of tropical basis for an ideal (see [MS15, Definition 2.6.4] for the definition of tropical basis).

The discussion above motivates the following question:

Question 1.5. *Which linear differential polynomials $F \in K[\![t]\!]\{x\}$ admit a* finite *differential tropical basis for $[F]$?*

The matter is subtle: [FT22, Section 4.2] shows that, generically, a second-order linear ODE does not have a finite DTB of linear forms (this does not rule out the existence of a finite tropical basis of higher degree, *a priori*). On the other hand, in Subsection 2.2 we identify two sufficient conditions on the equation for a finite DTB of linear forms to exist. In the spirit of Question 1.2, one can ask:

Question 1.6. *Can we define a notion of differential tropical basis that incorporates an additional group action, beyond the differential one, so that all linear ODEs have a finite differential tropical basis of linear forms?*

2. Second order

2.1. **Trivial valuation.** Consider Equation (1.1) for the case $r = 2$. Let α and β denote the roots of the characteristic polynomial $\chi(z)$. Assuming $\alpha \neq \beta$, the matroid M is represented by the matrix

$$A = \begin{pmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \dots \\ 1 & \beta & \beta^2 & \beta^3 & \beta^4 & \dots \end{pmatrix}$$

If all maximal minors are non-zero, M is the infinite uniform matroid $U_{2,\mathbb{N}}$. Otherwise, we choose indices $k < l$ such that the minor formed by the k-th and l-th columns vanishes, and the difference $l - k$ is minimal. Factoring out the lowest powers yields:

$$\begin{vmatrix} \alpha^k & \alpha^l \\ \beta^k & \beta^l \end{vmatrix} = \alpha^k \beta^k \begin{vmatrix} 1 & \alpha^{l-k} \\ 1 & \beta^{l-k} \end{vmatrix} = \alpha^k \beta^k (\beta^{l-k} - \alpha^{l-k}) = 0.$$

Recall that α, β are non-zero, thus $\beta^{l-k} - \alpha^{l-k} = 0$ means that $\omega = \beta/\alpha$ is a $(l-k)$-th root of unity. So $\beta = \omega\alpha$ and we rescale the columns of A to obtain the following matrix

$$\tilde{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \dots & \omega^{l-k-1} & 1 & \omega & \dots \end{pmatrix}$$

Consequently, a set $\{a, b\}$ is a base if and only if $a \not\equiv b \mod (l-k)$. The base polytope P_M associated with M possesses infinitely many vertices $\mathbf{e}_{\{a,b\}}$; however, P_M is determined by the polytope associated with the finite uniform matroid $U_{2,l-k}$ plus an action on $\mathbb{R}^{\mathbb{N}}$ by the permutation group $H \subset S_{\mathbb{N}}$, generated by all the transpositions of the form (ij) satisfying $i \equiv j \mod (l-k)$. The number of orbits is finite, having canonical representatives $\{\bar{a}, \bar{b}\}$ where $\bar{a}, \bar{b} \in \{0, \dots, l-k-1\}$.

Regarding the flats: proper flats have rank 1 and there are $(l-k)$-many of them, namely the $l-k$ equivalence classes of $\mathbb{N}$ modulo $l-k$. Thus, remarkably, the Bergman fan of M consists of finitely many cones, despite the infinite ground set. See Figure 2.1 below.

It is natural at this point to ask about the Boolean dimension of the tropicalization of the solution set: this is the cardinality of a minimal system of generators. Interpreting the members of the tropical solution set as indicator vectors of subsets of $\mathbb{N}$, the Boolean dimension counts the number of inclusion minimal sets, thus it can be infinite. In this example, we have

$$\dim_{\mathbb{B}}(\mathrm{Trop}(\text{row-span}\, S)) = l - k,$$

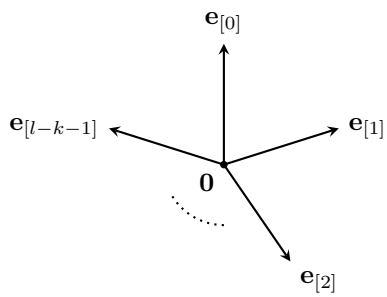

FIGURE 1. The Bergman fan of M. Despite the infinite ground set, the fan consists of exactly $l-k$ rays corresponding to the equivalence classes of $\mathbb{N}$ (mod $l-k$).

as the support of the generators is disjoint. Note that classically the dimension is 2, thus showing that dimension can increase arbitrarily under tropicalization.

Last, if $\alpha = \beta$, then the solutions are of the form

$$\exp(\alpha t), \quad t\exp(\alpha t),$$

and so we get the matrix:

$$S = \begin{pmatrix} 1 & \alpha & \frac{1}{2}\alpha^2 & \frac{1}{3}\alpha^3 & \frac{1}{4}\alpha^4 & \dots \\ 0 & 1 & \alpha & \alpha^2 & \alpha^3 & \dots \end{pmatrix}.$$

It is straightforward to see that this corresponds to the infinite uniform matroid $U_{2,\mathbb{N}}$.

2.2. **p-adic valuation.** We revisit the example from [FT22, Section 4.2], which demonstrates that a generic second-order linear equation with constant coefficients lacks a finite DTB of linear forms in the trivially valued case. Shifting to the non-trivially valued case we show that if the roots of the characteristic polynomial have distinct valuations, a finite DTB is guaranteed to exist.

Let $(\mathbb{C}_p, \nu_p)$ denote the valued field of p-adic complex numbers for a prime number p, and consider the differential polynomial

$$F = (x-\alpha_1)(x-\alpha_2) = x^2 - (\alpha_1+\alpha_2)x + \alpha_1\alpha_2 \in \mathbb{C}_p[\![t]\!]\{x\}.$$

As recalled above, the classical solutions are $\exp(\alpha_1 t)$ and $\exp(\alpha_2 t)$. Let $u_1 = \nu_p(\alpha_1)$ and $u_2 = \nu_p(\alpha_2)$ denote the valuations of the roots and set $c = p$ as its customary for the p-adic norm. It follows from the fact that the radius of convergence of the p-adic exponential is $p^{-\frac{1}{p-1}}$ that the radius of convergence of these two series is:

$$\rho(\exp(\alpha_1 t)) = p^{-\frac{1}{p-1}+u_1}$$
$$\rho(\exp(\alpha_2 t)) = p^{-\frac{1}{p-1}+u_2}$$

Assuming $u_1 < u_2$, the properties of valuations imply that $\alpha_1 \neq 0$ and $\nu_p(\alpha_1+\alpha_2) = u_1$, with the case $\alpha_2 = 0$ corresponding to $u_2 = \infty$.

We aim to characterize the tropical power series $B = \sum_{i\in\mathbb{N}} b_i t^i \in \mathbb{T}[\![t]\!]_{\nu_p}$ solving the system $\{\mathrm{trop}(F^{(n)}) \mid n \in \mathbb{N}\}$. Since F has order $r = 2$, the initial coefficients b_0 and b_1 completely determine the series. By rescaling the solution, we may assume without loss of generality

that $b_0 = 0$, and treat b_1 as a free parameter to compute the subsequent terms. Evaluating $\mathrm{trop}(F^{(n)})$ at B yields the following tropical vanishing condition for all $n \in \mathbb{N}$:

$$(0, b_n + u_1 + u_2 + \nu_p(n!)) \oplus (0, b_{n+1} + u_1 + \nu_p((n+1)!)) \oplus (0, b_{n+2} + \nu_p((n+2)!)). \tag{2.1}$$

If $u_2 = \infty$, then Equation (2.1) reduces to the vanishing of

$$(0, b_{n+1} + u_1 + \nu_p((n+1)!)) \oplus (0, b_{n+2} + \nu_p((n+2)!)). \tag{2.2}$$

This immediately yields the recurrence relation $b_{n+2} = b_{n+1} + u_1 - \nu_p(n+2)$ for all $n \geq 1$.

It remains to consider the case where $u_2 < \infty$. Setting $n = 0$ in Equation (2.1) requires the following expression to tropically vanish:

$$(0, u_1 + u_2) \oplus (0, b_1 + u_1) \oplus (0, b_2 + \nu_p(2!)).$$

This is equivalent to defining the tropical polynomial

$$G_0 = \min(u_1 + u_2, u_1 + b_1, b_2 + \nu_p(2!)) \in \mathbb{T}[b_1, b_2],$$

and requiring the point (b_1, b_2) to lie in the tropical variety $V^{\mathrm{trop}}(G_0) \subset \mathbb{T}^2$. In other words, the minimum in G_0 must be achieved at least twice. This requirement yields three distinct cases, depending on the value of the parameter b_1. In the following boxed equations we follow the first case as it branches along.

(1) if $b_1 < u_2$: the minimum value is $b_1 + u_1$, achieved by the second and third terms of G_0; forcing $\boxed{b_2 = b_1 + u_1 - \nu_p(2!)}$.
(2) if $b_1 > u_2$: the minimum is $u_1 + u_2$, achieved by the first and third terms; forcing $b_2 = u_1 + u_2 - \nu_p(2!)$.
(3) if $b_1 = u_2$: the first two terms coincide: $u_1 + u_2 = b_1 + u_1$. So G_0 vanishes tropically if and only if $b_2 \geq u_1 + u_2 - \nu_p(2!)$.

Next, analogously as above, setting $n = 1$ in Equation (2.1) yields:

$$G_1 = \min(b_1 + u_1 + u_2, b_2 + u_1 + v_p(2!), b_3 + \nu_p(3!)) \in \mathbb{T}[b_2, b_3].$$

To compute the value of b_3 that makes G_1 tropically vanish, the cases are as follows:

(1) if $b_1 < u_2$: recall that $b_2 = b_1 + u_1 - v_p(2!)$, so $G_1 = \min(b_1 + u_1 + u_2, b_1 + 2u_1, b_3 + \nu_p(3!))$. As $u_1 < u_2$, the minimum is $b_1 + 2u_1$, achieved by the second and third terms of G_1; forcing $\boxed{b_3 = b_1 + 2u_1 - \nu_p(3!)}$.
(2) if $b_1 > u_2$: recall that $b_2 = u_1 + u_2 - v_p(2!)$, so $G_1 = \min(b_1 + u_1 + u_2, u_2 + 2u_1, b_3 + \nu_p(3!))$. As $u_1 < u_2 < b_1$, the minimum is $u_2 + 2u_1$, achieved by the second and third terms of G_1; forcing $b_3 = u_2 + 2u_1 - \nu_p(3!)$.
(3) $b_1 = u_2$: recall that $u_1 + u_2 = b_1 + u_1$, so $G_1 = \min(u_1 + 2u_2, b_2 + u_1 + v_p(2!), b_3 + \nu_p(3!))$. Also recall that $b_2 \geq u_1 + u_2 - \nu_p(2!)$, so the second term is not smaller than the first term. Now we are in an identical position as the $n = 0$ case, yielding three sub-cases:
   (3.1) $b_2 < 2u_2 - v_p(2!)$: proceeds analogously to case (1). From $b_2 < 2u_2 - v_p(2!)$ we get $b_2 + v_p(2!) + u_1 < 2u_2 + u_1$ forcing $b_3 = b_2 + \nu_p(2!) + u_1 - \nu_p(3!)$.
   (3.2) $b_2 > 2u_2 - v_p(2!)$: proceeds analogously to case (2). From $b_2 > 2u_2 - v_p(2!)$ we get $b_2 + u_1 + v_p(2!) > 2u_2 + u_1$ forcing $b_3 = 2u_2 + u_1 - v_p(3!)$.
   (3.3) $b_2 = 2u_2 - v_p(2!)$: We obtain $G_1 = \min(u_1 + 2u_2, u_1 + 2u_2, b_3 + \nu_p(3!))$. The only constraint on b_3 is the inequality $b_3 \geq u_1 + 2u_2 - \nu_p(3!)$.

By induction, one can prove that in cases 1 and 2:

(i) Case $b_1 < u_2$: assume $b_k = b_1 + (k-1)u_1 - \nu_p(k!)$. Evaluating, we have $G_{k-1} = \min(b_1 + (k-1)u_1 + u_2, b_1 + ku_1, b_{k+1} + \nu_p((k+1)!))$. As $u_1 < u_2$, the second term is the unique minimum among the first two; thus G_{k-1} tropically vanishes if and only if $\boxed{b_{k+1} = b_1 + ku_1 - \nu_p((k+1)!)}$.

(ii) $b_1 > u_2$: similar reasoning leads to the closed-form $b_{k+1} = u_2 + ku_1 - \nu_p((k+1)!)$ in this case.

Both of these families of solutions yield a tropical radius of convergence equal to $\rho(\exp(\alpha_1 t))$.

Case 3 exhibits unstable behaviour. On the one hand, at each step one can keep choosing $b_i = iu_2 - \nu_p(i!)$ to build a solution with a tropical radius of convergence equal to $\rho(\exp(\alpha_2 t))$. On the other hand, at each step the parameter b_i only needs to satisfy the constrain $b_i \geq u_1 + (i-1)u_2 - \nu_p(i!)$, thus a choice distinct than $b_i = iu_2 - \nu_p(i!)$ reverts the sequence to a solution with a radius of convergence $\rho(\exp(\alpha_1 t))$

These branching solutions are fully determined by a branching index $n \in \mathbb{N} \cup \{\infty\}$ and a parameter $b \in \mathbb{T}$ such that $b \geq u_1 + (n-1)u_2 - \nu_p(n!)$ when $n > 1$. Specifically, a sequence that eventually behaves as in Case 1, namely $b < nu_2 - \nu_p(n!)$, is given by

$$\begin{aligned} b_i &= iu_2 - \nu_p(i!) \text{ for all } i < n, \\ b_n &= b, \\ b_{n+i} &= b + \nu_p(n!) + iu_1 - \nu_p((n+i)!) \text{ for all } n+i > n. \end{aligned}$$

A sequence that eventually behaves as in Case 2, namely $b > nu_2 - \nu_p(n!)$, is given by:

$$\begin{aligned} b_i &= iu_2 - \nu_p(i!) \text{ for all } i < n, \\ b_n &= b, \\ b_{n+i} &= nu_2 + \nu_p(n!) + iu_1 - \nu_p((n+i)!) \text{ for all } n+i > n. \end{aligned}$$

Moreover, all the solutions we have described are realizable, that is, they are the tropicalizations of classical solutions. Indeed, consider the following convenient elements:

$$\begin{aligned} \psi_n =& \frac{n!p^b}{\alpha_1^n} \exp(\alpha_1 t) \\ =& \frac{n!p^b}{\alpha_1^n} + \frac{n!p^b}{\alpha_1^{n-1}} t + \frac{1}{2} \frac{n!p^b}{\alpha_1^{n-2}} t^2 + \dots \\ \varphi_n =& \frac{\exp(\alpha_1 t)}{\alpha_1^n} - \frac{\exp(\alpha_2 t)}{\alpha_2^n} = \sum_{i=0}^{\infty} \frac{1}{i!} (\alpha_1^{i-n} - \alpha_2^{i-n}) t^i \\ =& \left(\frac{1}{\alpha_1^n} - \frac{1}{\alpha_2^n} \right) + \left(\frac{1}{\alpha_1^{n-1}} - \frac{1}{\alpha_2^{n-1}} \right) t + \dots + \frac{1}{(n-1)!} \left(\frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right) t^{n-1} \\ &+ \frac{1}{(n+1)!} (\alpha_1 - \alpha_2) t^{n+1} + \dots \end{aligned}$$

We write $[A]_i$ for the i-th term of the series A. Recall $b + \nu_p(n!) \geq u_1 + (n-1)u_2$, thus:

$$\begin{aligned}[\mathrm{trop}(\psi_n)]_i &= b + \nu_p(n!) + (i-n)u_1 - \nu_p(i!) \\ &\geq iu_2 + (n-i-1)(u_2 - u_1) - \nu_p(i!)\end{aligned}$$

$$[\mathrm{trop}(p^{nu_2}\varphi_n)]_i = \begin{cases} iu_2 - \nu_p(i!) & \text{if } i < n, \\ \infty & \text{if } i = n, \\ nu_2 + (i-n)u_1 - \nu_p(i!) & \text{if } i > n. \end{cases}$$

Note above that $(u_2 - u_1)$ is positive. Given parameters n and b, we have two cases. If $b < nu_2 - \nu_p(n!)$, there is a generic element ϵ of valuation 0 such that

$$\begin{aligned}[\mathrm{trop}(\exp(\alpha_2 t) + \epsilon\psi_n)]_i &= \min(iu_2 - \nu_p(i!), b + \nu_p(n!) + (i-n)u_1 - \nu_p(i!)) \\ &= \begin{cases} iu_2 - \nu_p(i!) & \text{if } i < n, \\ b & \text{if } i = n, \\ b + \nu_p(n!) + (i-n)u_1 - \nu_p(i!) & \text{if } i > n. \end{cases}\end{aligned}$$

If $b \geq nu_2 - \nu_p(n!)$, set $b' = b - (nu_2 - \nu_p(n!)) \geq 0$. Then, again up to multiplying by a generic element ϵ of valuation 0, we can realize a branching that eventually behaves as Case 2 by:

$$[\mathrm{trop}(p^{b'}\exp(\alpha_2 t) + \epsilon p^{nu_2}\varphi_n)]_i = \begin{cases} iu_2 - \nu_p(i!) & \text{if } i < n, \\ b & \text{if } i = n, \\ nu_2 + (i-n)u_1 - \nu_p(i!) & \text{if } i > n. \end{cases}$$

Finally, $\exp(\alpha_2 t)$ realizes the unstable solution. This shows that $\{F\}$ is a DTB for the ideal $[F]$. On the other hand, the following question remains open:

Question 2.1. *What are the possible behaviours when the roots α_1, α_2 have the same valuation? Is there an ideal $[F]$ with a finite DTB that consists of more than one element?*

3. Order 3

3.1. **Trivial valuation.** Now we consider Equation (1.1) for $r = 3$. Let α, β, γ denote the roots of the characteristic polynomial $\chi(z)$. Assuming the roots are distinct, the matroid M is represented by the matrix

$$S = \begin{pmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \dots \\ 1 & \beta & \beta^2 & \beta^3 & \beta^4 & \dots \\ 1 & \gamma & \gamma^2 & \gamma^3 & \gamma^4 & \dots \end{pmatrix}$$

Assume that we do not get the uniform matroid $U_{3,\mathbb{N}}$. We choose indices $k < l < m$ of a vanishing minor. Via row and column operations it can be shown that the vanishing of the minor of $S_{k,l,m}$ is equivalent to the vanishing of

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & (\beta/\alpha)^{l-k} - 1 & (\beta/\alpha)^{m-k} - 1 \\ 0 & (\gamma/\alpha)^{l-k} - 1 & (\gamma/\alpha)^{m-k} - 1 \end{vmatrix} = ((\beta/\alpha)^{l-k} - 1)((\gamma/\alpha)^{m-k} - 1) - ((\beta/\alpha)^{m-k} - 1)((\gamma/\alpha)^{l-k} - 1).$$

We consider the rational function

$$f_{\mu,\eta}(w) = \frac{w^\mu - 1}{w^\eta - 1}. \tag{3.1}$$

The vanishing of Equation (3.1) is equivalent to the equality

$$f_{m-k,l-k}(\gamma/\alpha) = f_{m-k,l-k}(\beta/\alpha).$$

Note that besides α, β, γ being distinct from zero, the fact that they are all pairwise distinct means that neither the ratio γ/α nor β/α can be 1.

If both γ/α and β/α are roots of unity, then we get again a periodic matrix S. Otherwise, we pose the following question:

Question 3.1. *Under trivial valuation, is having a tropical basis, in the sense of Fink and Toghani, equivalent to having an associated polyhedral complex with finitely many cones?*

We have seen in Subsection 2.2 that the assumption of trivial valuation is needed to pose the question, otherwise roots with different valuation give rise to infinitely many polyhedral cells.

3.2. **p-adic valuation.** Following a similar strategy to that of Subsection 2.2 one can conclude the following. We label the roots of the characteristic polynomial as $\alpha_1, \alpha_2, \alpha_3$ with $\nu(\alpha_1) \leq \nu(\alpha_2) \leq \nu(\alpha_3)$, and let $u_i = \nu(\alpha_i)$. In this case we can normalize $b_0 = 0$ as before, and search conditions for when the parameters (b_1, b_2) determine the solution. It turns out that if we assume that $\nu(\alpha_1)$ is strictly less than u_2, u_3, then (b_1, b_2) determine the solution if they are in the complement of the tropical line defined by $\min(u_2 + u_3, x + u_2, y)$. Moreover, if all u_i are distinct we can conclude that $\{F\}$ is a tropical bases, using similar methods for crafting the tropical solution, and Cramer's rule for showing realizability. The details of this computation will be included in upcoming work.

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Email address: stefano.mereta@cunef.edu

Departamento de Matemáticas, CUNEF Universidad, C. de Almansa 101, 28040 Madrid, Spain

Email address: alejandro@vargas.page

Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, England, UK